Math 155, ***Lecture Notes- Bonds*** **Name**____________

Section 9.8 *Power Series*

In this section, we will learn that several types of important functions can be represented *exactly* by infinite series called **power series**.
For example,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots$$

Eventually, we will see that for each real number x, the infinite series on the right side will converge to the number e^x.

Definition of Power Series

If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

is called a **power series.** More generally, an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n + \cdots$$

is called a **power series centered at** c**,** where c is a constant.

$n=0 \quad n=1 \quad n=2 \quad n=3 \quad n=4$

Ex. 1: (a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1} = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} + \cdots$

center is 0.

$n=1 \quad n=2 \quad n=3 \quad n=4$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^3} = \frac{(x-2)^1}{1} + \frac{(x-2)^2}{2^3} + \frac{(x-2)^3}{3^3} + \frac{(x-2)^4}{4^3} + \cdots$

$\displaystyle = (x-2) + \frac{(x-2)^2}{8} + \frac{(x-2)^3}{27} + \frac{(x-2)^4}{64} + \cdots$

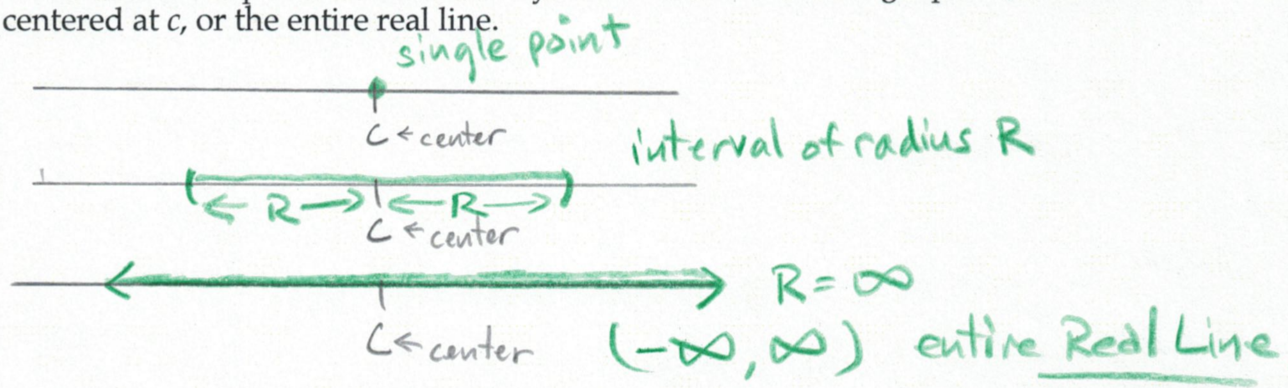
center is 2.

We can view $f(x)=\sum_{n=0}^{\infty} a_n (x-c)^n$ as a function of x where the domain of f is the set of all x for which the power series converges. Therefore, we will need to know the values of x that allow the series to converge, and determination of this domain will be the main focus of this section.

First, every power series converges at its center c:

$$
\begin{aligned}
f(c) &= \sum_{n=0}^{\infty} a_n (c-c)^n \\
&= a_0(1)+0+0+0\cdots \quad \text{, where we are agreeing that } (x-c)^0=1, \text{ even if } x=c. \\
&= a_0
\end{aligned}
$$

The domain of a power series has only three basic forms: a single point, an interval centered at c, or the entire real line.



THEOREM 9.20 Convergence of a Power Series

For a power series centered at c, precisely one of the following is true.

1. The series converges only at c.
2. There exists a real number $R > 0$ such that the series converges absolutely for $|x - c| < R$, and diverges for $|x - c| > R$.
3. The series converges absolutely for all x.

The number R is the **radius of convergence** of the power series. If the series converges only at c, the radius of convergence is $R = 0$, and if the series converges for all x, the radius of convergence is $R = \infty$. The set of all values of x for which the power series converges is the **interval of convergence** of the power series.

Ex. 2: Determine the interval of convergence of the series: $\sum_{n=0}^{\infty} \frac{(3x)^n}{(2n)!}$

When we are looking for a radius of convergence, we are considering absolute convergence. We can use the Ratio Test to find an interval for absolute convergence. Let's calculate R, the radius of convergence.

Let $\sum_{n=0}^{\infty} \frac{(3x)^n}{(2n)!} = \sum_{n=0}^{\infty} u_n$, with $u_n = \frac{(3x)^n}{(2n)!}$. The center of this power series is zero, and we know all power series are convergent at their centers. So, this series converges for $x=0$. Since we are going to use the Ratio Test, we need to see that $u_n = \frac{(3x)^n}{(2n)!} \neq 0$. So, we will not consider $x=0$.

Consider $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(3x)^{n+1}}{[2(n+1)]!}}{\frac{(3x)^n}{(2n)!}}\right|$

$= \lim_{n\to\infty} \left|\frac{(3x)^{n+1}}{[2n+2]!} \cdot \frac{(2n)!}{(3x)^n}\right|$

$= \lim_{n\to\infty} \left|\frac{(3x)\cancel{(3x)^n} \cdot \cancel{(2n)!}}{(2n+2)(2n+1)\cdot\cancel{(2n)!}\cdot\cancel{(3x)^n}}\right|$

$= \lim_{n\to\infty} \left|\frac{(3x)}{(2n+2)(2n+1)}\right|$

$= |3x| \cdot \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)}$

$= |3x| \cdot 0$

$= 0$. Since $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$, the Ratio Test tells us that $\sum_{n=0}^{\infty} \frac{(3x)^n}{(2n)!}$ converges absolutely for all x, and that $R=\infty$.

This means that the interval of convergence is $(-\infty, \infty)$.

Note that for a power series with a radius of convergence that is a finite number R, Theorem 9.20 says nothing about the convergence at the *endpoints* of the interval of convergence. In fact, each endpoint must be tested separately for convergence or divergence.

Ex. 3: Determine the interval of convergence of the series: $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

We can use the Ratio Test to find an interval for absolute convergence, but we might need to check for convergence at the endpoints of our interval.

Let $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} u_n$, where $u_n = \frac{(-1)^n x^{2n+1}}{2n+1}$.

The center of this series is zero, and we know all power series are convergent at their centers. So, this series converges for $x=0$. Since we are going to use the Ratio Test, we need to see that $u_n = \frac{(-1)^n x^{2n+1}}{2n+1} \neq 0$. So, we will not consider $x=0$.

Consider $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} x^{2(n+1)+1}}{2(n+1)+1}}{\frac{(-1)^n x^{2n+1}}{2n+1}} \right|$

$$= \lim_{n\to\infty} \left| \frac{x^{2n+3}}{2n+3} \cdot \frac{2n+1}{x^{2n+1}} \right|$$

$$= \lim_{n\to\infty} |x^2| \cdot \left(\frac{2n+1}{2n+3}\right)$$

$$= |x^2| \cdot \lim_{n\to\infty} \left(\frac{\frac{2n+1}{1}}{\frac{2n+3}{1}}\right)\left(\frac{\frac{1}{2n}}{\frac{1}{2n}}\right)$$

$$= |x^2| \cdot \lim_{n\to\infty} \left(\frac{1+\frac{1}{2n}}{1+\frac{3}{2n}}\right)$$

$$= x^2 \cdot 1$$

$$= x^2$$

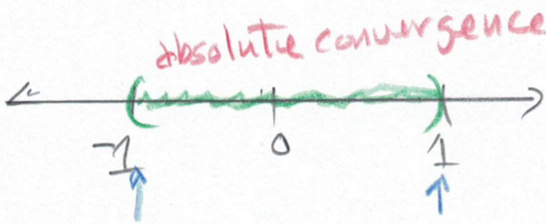
According to the Ratio Test, we have absolute convergence when $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$, or $x^2 < 1$,

More Ex. 3:

We can solve $x^2 < 1$ for x.

Let $x^2 = 1$

Either $x = 1$ or $x = -1$



Test	Test	Test
$x=-2$	$x=0$	$x=2$
$(-2)^2 < 1$	$(0)^2 < 1$	$(2)^2 < 1$
$4 < 1$	$0 < 1$	$4 < 1$
False	TRUE	False

$-1 < x < 1$

We have to check endpoint convergence

If $x = 1$, then $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n (1)^{2n+1}}{2n+1}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

We can use the Alternating Series Test to see that $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ converges. Let $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \cdot a_n$, where $a_n = \frac{1}{2n+1}$.

We know $1 > 0$ and $2n+1 > 0$ for all $n \geq 1$, and we can see that $\frac{1}{2n+1} > 0$ since a ratio of positive numbers is positive. (RoPNiP) This means that $a_n > 0$ for all $n \geq 1$.

Consider $\displaystyle\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{2n+1} = 0$.

We know $2n+1 \leq 2n+3$ for all $n \geq 1$. We can see

$$\frac{1}{(2n+1)(2n+3)} \cdot \left(\frac{2n+1}{1}\right) \leq \frac{1}{(2n+1)(2n+3)} \cdot \left(\frac{2n+3}{1}\right)$$

$$\frac{1}{2n+3} \leq \frac{1}{2n+1}$$

$$\frac{1}{2(n+1)+1} \leq \frac{1}{2n+1}$$

$$a_{n+1} \leq a_n$$

Therefore, according to the Alternating Series Test, $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ converges.

Still More Ex. 3:

If $x=-1$, then $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (-1)^{2n+1}}{2n+1}$ ← Always odd

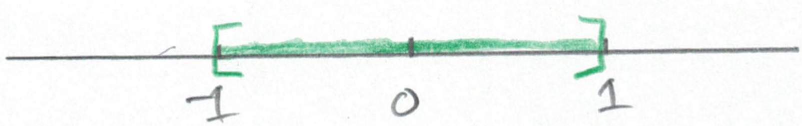
$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (-1)}{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$$

The Alternating Series Test can be used to show that the $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$ converges. Let $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^{n+1} a_n$, with $a_n = \frac{1}{2n+1}$.

Since $a_n = \frac{1}{2n+1}$ is the same as the one we considered in the previous Alternating Series Test, we have convergence according to the previous test.

This means that we have convergence at both endpoints, $x=-1$ and $x=1$, so the interval of convergence for the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ is $[-1, 1]$.



Ex. 4: Determine the interval of convergence of the series: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n2^n}$

We can use the Ratio Test to find an interval for absolute convergence, but we might need to check for convergence at the endpoints of our interval.

Let $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n2^n} = \sum_{n=1}^{\infty} u_n$, where $u_n = \frac{(-1)^{n+1}(x-2)^n}{n2^n}$.

The center of this series is $c=2$, and we know all power series are convergent at their centers. So, this series converges for $x=2$. Since we are going to use the Ratio Test, we need to see that $u_n = \frac{(-1)^{n+1}(x-2)^n}{n2^n} \neq 0$. So, we will not consider $x=2$.

Consider $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{(n+1)+1}(x-2)^{n+1}}{(n+1)2^{(n+1)}}}{\frac{(-1)^{n+1}(x-2)^n}{n2^n}}\right|$

$= \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}\cdot n2^n}{(x-2)^n\cdot(n+1)2^{n+1}}\right|$

$= \lim_{n\to\infty}\left|\frac{(x-2)\cdot n}{(n+1)\cdot 2}\right|$

$= \frac{|x-2|}{2}\cdot \lim_{n\to\infty}\frac{n}{n+1}$

$= \frac{|x-2|}{2}\cdot \lim_{n\to\infty}\left(\frac{\frac{n}{1}}{\frac{n+1}{1}}\right)\left(\frac{\frac{1}{n}}{\frac{1}{n}}\right)$

$= \frac{|x-2|}{2}\cdot \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}$

$= \frac{|x-2|}{2}\cdot 1$

$= \frac{|x-2|}{2}$

According to the Ratio Test, we have absolute convergence when $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$, or $\frac{|x-2|}{2} < 1$.

More Ex. 4:

We can solve $\frac{|x-2|}{2} < 1$ for x.

$$2 \cdot \frac{|x-2|}{2} < 2 \cdot 1$$

$$|x-2| < 2$$

$$-2 < x-2 < 2$$

$$-2+2 < x-2+2 < 2+2$$

$$0 < x < 4$$

We have to check endpoint convergence

If $x=4$, then $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n\,2^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}[(4)-2]^n}{n\,2^n}$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cancel{2^n}}{n\,\cancel{2^n}}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

We can use the Alternating Series Test to see that $\sum \frac{(-1)^{n+1}}{n}$ converges. Let $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} a_n$, where $a_n = \frac{1}{n}$.

We know $1>0$ and $n>0$ for all $n \geq 1$, and we can see that $\frac{1}{n} > 0$ since a ratio of positive numbers is positive. (RoPNiP)

This means that $a_n > 0$ for all $n \geq 1$.

Consider $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$.

Still More Ex. 4:

We know $n \le n+1$ for all $n \ge 1$. We can see

$$\frac{1}{n(n+1)} \cdot \left(\frac{n}{1}\right) \le \frac{1}{n(n+1)} \cdot \left(\frac{n+1}{1}\right)$$

$$\frac{1}{n+1} \le \frac{1}{n}$$

$a_{n+1} \le a_n$ Therefore, according to the Alternating Series Test, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

If $x=0$, then $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}[(0)-2]^n}{n2^n}$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot (-2)^n}{n2^n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^n \cdot \cancel{2^n}}{n\cancel{2^n}}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n}$$

Always odd.

$$= \sum_{n=1}^{\infty} \frac{-1}{n}$$

$$= -\sum_{n=1}^{\infty} \frac{1}{n}$$

This series diverges, since it is a harmonic series. Therefore, the interval of convergence is $(0,4]$, since we found convergence only at one endpoint, $x=4$.

THEOREM 9.21 Properties of Functions Defined by Power Series

If the function given by

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$
$$= a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

has a radius of convergence of $R > 0$, then, on the interval $(c - R, c + R)$, f is differentiable (and therefore continuous). Moreover, the derivative and antiderivative of f are as follows.

1. $$f'(x) = \sum_{n=1}^{\infty} na_n(x-c)^{n-1}$$
$$= a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots$$

2. $$\int f(x)\,dx = C + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1}$$
$$= C + a_0(x-c) + a_1\frac{(x-c)^2}{2} + a_2\frac{(x-c)^3}{3} + \cdots$$

The *radius of convergence* of the series obtained by differentiating or integrating a power series is the same as that of the original power series. The *interval of convergence*, however, may differ as a result of the behavior at the endpoints.

Ex. 5: Determine the interval of convergence for **(a)** $f(x)$, **(b)** $f'(x)$, **(c)** $f''(x)$, and **(d)** $\int f(x)dx$, given that $f(x) = \sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$.

First, let's find $f'(x)$, $f''(x)$, and $\int f(x)dx$. If $f(x) = \sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$,

then $f'(x) = \sum_{n=1}^{\infty} n\cdot\left(\frac{x}{2}\right)^{n-1}\cdot\left(\frac{1}{2}\right)$ by chain Rule

$$\boxed{f'(x) = \sum_{n=1}^{\infty} \frac{n}{2}\cdot\left(\frac{x}{2}\right)^{n-1}}$$

by chain Rule

$$f''(x) = \sum_{n=1}^{\infty} \frac{n}{2}\cdot\left[(n-1)\cdot\left(\frac{x}{2}\right)^{n-2}\cdot\left(\frac{1}{2}\right)\right]$$

$$\boxed{f''(x) = \sum_{n=1}^{\infty}\left(\frac{n}{2}\right)\left(\frac{n-1}{2}\right)\left(\frac{x}{2}\right)^{n-2}}$$

More Ex. 5:

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{n+1} \cdot (2)}{n+1}$$

Let $z = \frac{x}{2}$

$\frac{dz}{dx} = \frac{1}{2}$

$2\,dz = dx$

$$\boxed{\int f(x)\,dx = C + \sum_{n=0}^{\infty} \left(\frac{2}{n+1}\right)\left(\frac{x}{2}\right)^{n+1}}$$

The radius of convergence will be the same for all four functions. Only the endpoints of the interval of convergence may differ. We'll consider the endpoints after we find the radius of convergence using the Ratio Test to find the open interval that has absolute convergence for $f(x) = \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$.

Let $\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} u_n$, where $u_n = \left(\frac{x}{2}\right)^n$. The center of this power series is $c=0$, and the series converges at its center. We need to see $u_n = \left(\frac{x}{2}\right)^n \neq 0$, so we will not consider $x=0$. Consider $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{\left(\frac{x}{2}\right)^{n+1}}{\left(\frac{x}{2}\right)^n}\right|$

$$= \lim_{n\to\infty} \left|\frac{x}{2}\right|$$

$$= \left|\frac{x}{2}\right| \cdot \lim_{n\to\infty} 1$$

$$= \left|\frac{x}{2}\right| \cdot 1$$

$$= \left|\frac{x}{2}\right|$$

According to the Ratio Test, we have absolute convergence when $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$, or $\left|\frac{x}{2}\right| < 1$.

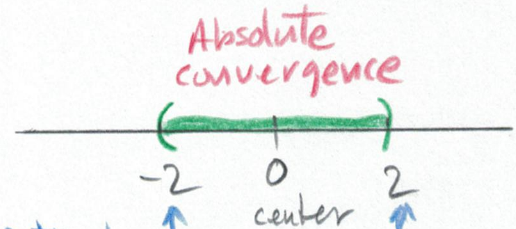
We can solve $\left|\frac{x}{2}\right| < 1$ for x.

$$2 \cdot \left|\frac{x}{2}\right| < 2 \cdot 1$$

$$|x| < 2$$

$$-2 < x < 2$$

$$R = 2$$



We have to check endpoint convergence

Still More Ex. 5:

Now that we have seen that the radius of convergence is $R = 2$ units, centered at $c = 0$, we check the endpoint convergence in the series $f(x)$, $f'(x)$, $f''(x)$, and $\int f(x)dx$.

(a) If $x = 2$, then $f(2) = \sum_{n=0}^{\infty} \left(\frac{2}{2}\right)^n = \sum_{n=0}^{\infty} (1)^n$, and we have a divergent series according to the n-th Term Test for Divergence.

If $x = -2$, then $f(-2) = \sum_{n=0}^{\infty} \left(\frac{-2}{2}\right)^n = \sum_{n=0}^{\infty} (-1)^n$, and we also have a divergent series according to the n-th Term Test for Divergence.

$(-2, 2)$

(b) If $x = 2$, then $f'(2) = \sum_{n=1}^{\infty} \frac{n}{2}\left(\frac{2}{2}\right)^{n-1} = \sum_{n=1}^{\infty} \frac{n}{2}(1)^{n-1} = \sum_{n=1}^{\infty} \frac{n}{2}$, and we have a divergent series according to the n-th Term Test for Divergence.

If $x = -2$, then $f'(-2) = \sum_{n=1}^{\infty} \frac{n}{2}\left(\frac{-2}{2}\right)^{n-1} = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{n}{2}$, and we also have a divergent series according to the n-th Term Test for Divergence.

$(-2, 2)$

(c) If $x = 2$, then $f''(2) = \sum_{n=1}^{\infty} \left(\frac{n}{2}\right)\left(\frac{n-1}{2}\right)\left(\frac{2}{2}\right)^{n-2} = \frac{1}{4}\sum_{n=1}^{\infty} (n^2 - n)$, and we have a divergent series according to the n-th Term Test for Divergence.

If $x = -2$, then $f''(-2) = \sum_{n=1}^{\infty} \left(\frac{n}{2}\right)\left(\frac{n-1}{2}\right)\left(\frac{-2}{2}\right)^{n-2} = \frac{1}{4}\sum_{n=1}^{\infty} (-1)^{n-2}(n^2 - n)$, and we also have a divergent series according to the n-th Term Test for Divergence.

$(-2, 2)$

(d) If $x = 2$, then we can evaluate $\int f(x)dx = C + \sum_{n=0}^{\infty} \left(\frac{2}{n+1}\right)\left(\frac{x}{2}\right)^{n+1}$ and check: $\sum_{n=0}^{\infty} \left(\frac{2}{n+1}\right)\left(\frac{2}{2}\right)^{n+1} = 2\sum_{n=1}^{\infty} \frac{1}{n+1}$, and we have a divergent harmonic series.

If $x = -2$, then we can evaluate $\int f(x)dx = C + \sum_{n=0}^{\infty} \left(\frac{2}{n+1}\right)\left(\frac{x}{2}\right)^{n+1}$ and check: $\sum_{n=0}^{\infty} \left(\frac{2}{n+1}\right)\left(\frac{-2}{2}\right)^{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}2}{n+1}$, and we have a convergent series, according to the Alternating Series Test.

$[-2, 2)$

Ex. 6: Given $f(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}$.

(a) Find the interval of convergence.

(b) Show that $f'(x)=f(x)$.

(c) Show that $f(0)=1$.

(d) Identify the function.

(a) We can use the Ratio Test to find the interval of convergence, but we might need to check for convergence at the endpoints of our interval.

Let $\sum_{n=0}^{\infty}\frac{x^n}{n!}=\sum_{n=0}^{\infty}u_n$, where $u_n=\frac{x^n}{n!}$. The center of this series is $c=0$ and we know all power series converge at their centers. So, this series converges at $x=0$. Since we are going to use the Ratio Test, we need to see that $u_n=\frac{x^n}{n!}\neq 0$. So, we will not consider $x=0$.

Consider
$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right|$$
$$=\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right|$$
$$=\lim_{n\to\infty}\left|\frac{x\cdot\cancel{x^n}\cdot\cancel{n!}}{(n+1)\cdot\cancel{n!}\cdot\cancel{x^n}}\right|$$
$$=\lim_{n\to\infty}\left|\frac{x}{n+1}\right|$$
$$=|x|\cdot\lim_{n\to\infty}\frac{1}{n+1}$$
$$=|x|\cdot 0$$
$$=0$$

Since $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|<1$, the Ratio Test tells us that $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ converges absolutely for all x, and that $R=\infty$.

This means that the interval of convergence is $(-\infty,\infty)$.

More Ex. 6:

(b) Show that $f'(x) = f(x)$. If $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, then $f'(x) = \sum_{n=1}^{\infty} \frac{1}{n!} \cdot (n \cdot x^{n-1})$.

$$f'(x) = \sum_{n=1}^{\infty} \frac{n \cdot x^{n-1}}{n \cdot (n-1)!}$$

$$f'(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$$, Let $k = n-1$, and we have $k = 0$ when $n = 1$, since $k = (1) - 1 = 0$.

$$f'(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$f'(x) = f(x) \checkmark$$

(c) Show that $f(0) = 1$.

If $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$$f(x) = \frac{1}{1} + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

and $f(0) = 1 + (0) + \frac{(0)^2}{2} + \frac{(0)^3}{3!} + \frac{(0)^4}{4!} + \cdots$

$$f(0) = 1 \checkmark$$

(d) Identify the function, $f(x)$. Assume $f(x) \neq 0$.

Since $f'(x) = f(x)$, we can see $\frac{f'(x)}{f(x)} = 1$, and

$$\int \left(\frac{f'(x)}{f(x)}\right) dx = \int (1)\, dx$$

$$\int \frac{f'(x)}{f(x)}\, dx = x + C$$

$$\int \frac{f'(x)}{u} \cdot \left(\frac{du}{f'(x)}\right) = x + C$$

$$\int \frac{1}{u}\, du = x + C$$

$$\ln|u| = x + C$$

$$\ln|f(x)| = x + C$$

Let $u = f(x)$

$$\frac{du}{dx} = f'(x)$$

$$du = f'(x)\, dx$$

$$\frac{du}{f'(x)} = dx$$

still More Ex. 6:

$$e^{\ln|f(x)|} = e^{x+c}$$

$$|f(x)| = e^x \cdot e^c$$

$$f(x) = \pm e^c \cdot e^x.$$ Let $D = \pm e^c$, so we have

$$f(x) = De^x$$

We know if $x=0$, then $f(0)=1$, and $f(0) = De^{(0)}$.

So, $1 = D \cdot 1 = D$.

Therefore, we have $f(x) = 1 \cdot e^x$ and $f(x) = e^x$.

This means $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.